

Stability and anyonic behavior of systems with M -statistics

Marcelo R. Ubriaco*

Laboratory of Theoretical Physics
Department of Physics
University of Puerto Rico
Río Piedras Campus
San Juan
PR 00931, USA

Abstract

Starting with the partition function Z for systems with M -statistics, as proposed in [1], we calculate from the metric $g_{\alpha\eta} = \frac{\partial \ln Z}{\partial \beta^\alpha \partial \beta^\eta}$ the scalar curvature R in two and three dimensions. Our results exhibit the details of the anyonic behavior as a function of the fugacity z and the identical particle maximum occupancy number M . We also compare the stability of systems for $M > 1$ with the fermionic ($M = 1$), and bosonic ($M \rightarrow \infty$), cases.

PACS numbers: 05.30.-d, 02.40.-k, 05.30.Pr

1 Introduction

Several years ago a new fractional statistics was proposed [1] in arbitrary dimensions which allows a finite number of multi-occupancy states of a single state. Since that a single quantum state can be occupied by $n \leq M$ identical particles, this formalism opens the way to a new thermodynamics based on an extension of the Pauli exclusion principle. From a calculation of the second virial coefficient for low values of the fugacity z it was found in [1] that an ideal gas becomes bosonic for $M > 1$. Also, a calculation of the heat capacity C_V for high z values showed that the system remains fermionic for $M > 1$. Certainly, these approximate calculations show that a thermodynamic system that follows M -statistics exhibit anyonic behavior for low z values but miss the details that could tell us how this behavior depends on the fugacity z and the maximum occupancy M . In addition, one important question that one may address is related to the

*Electronic address: ubriaco@ltp.uprrp.edu

stability of systems based on M -statistics as compared with fermionic ($M = 1$) and bosonic ($M \rightarrow \infty$) cases. The answer to these questions can be given by performing a calculation of the thermodynamic curvature R , which in our case, is nothing else than the scalar curvature in the two-dimensional space defined by the parameters β and $\gamma = -\beta\mu$. The idea of using geometry to study some properties of thermodynamic systems [2]-[6] opened the way to the basic formalism of defining a metric in a two dimensional parameter space and calculate the corresponding scalar curvature as a measure of the correlations strength of the system [8]-[17], with applications to classical and quantum gases [7][13][18][19], magnetic systems [20]-[23], non-extensive statistical mechanics [24]-[26], anyon gas [27]-[28], fractional statistics [29], deformed boson and fermion systems [30], systems with fractal distribution functions [31] and quantum group invariance [32]. Some of the basic results of this formalism is the relationship between the departure of the scalar curvature R from the zero value and the stability of the system, and the fact that R vanishes for a classical gas, $R > 0$ ($R < 0$) for a boson (fermion) gas and becomes singular at a critical point. Here, in order to study the stability and anyonic behavior of systems with M -statistics we calculate the dependence of the scalar curvature on the fugacity z and the maximum occupation number M . In Section 2 we briefly describe the formalism of M -statistics. In Section 3 we calculate the scalar curvature for M -statistics in two and three dimensions. In Section 4 we discuss and summarize our results.

2 Quantum M-statistics

In this section we briefly introduce the formalism of M -statistics, the details and the comparison of M -statistics with fractional exclusion statistics can be found in Ref. [1]. For a single species of particles the set of orthonormal eigenstates of the number operator \hat{N} are denoted by $|j\rangle$ where $j = 0, 1, 2, \dots, M$ and

$$\hat{N}|j\rangle = j|j\rangle. \quad (1)$$

The annihilation and creation operators a and a^\dagger satisfy a set of relations

$$(aa^\dagger \pm a^\dagger a)|j\rangle = (|f_{j+1}|^2 \pm |f_j|^2)|j\rangle, \quad (2)$$

where

$$a|j\rangle = f_j|j-1\rangle, \quad a^\dagger|j\rangle = f_{j+1}^*|j+1\rangle. \quad (3)$$

In particular, $f_0 = 0$ and $f_1 = e^{i\alpha}$. It is clear that $a^j = a^{\dagger j} = 0$ for all values of j such that $j > M$. For multi-species of particles the states are simply written as $|ijk\dots\rangle$ with the corresponding number operators satisfying commutation relations $[\hat{N}_i, \hat{N}_j] = 0$ and the a and a^\dagger operators satisfying two possible commutation rules

$$\begin{aligned} a_i a_j &= e^{\pm i \frac{2\pi}{M+1}} a_j a_i = 0, \\ a_i^\dagger a_j &= e^{\mp i \frac{2\pi}{M+1}} a_j a_i^\dagger = 0 \quad i < j. \end{aligned} \quad (4)$$

It is simple to see that the states are fermionic for $M = 1$, and become bosonic for $M \rightarrow \infty$. The partition function takes the simple form

$$Z = \prod_p \sum_{j=0}^M z^j e^{-j\beta\epsilon(p)}, \quad (5)$$

In the thermodynamic limit, we obtain in arbitrary dimensions D

$$\ln Z = VA(D) \int_0^\infty dp p^{D-1} \ln \frac{1 - z^{M+1} e^{-(M+1)\beta\epsilon(p)}}{1 - ze^{-\beta\epsilon(p)}}, \quad (6)$$

with $\epsilon(p) = p^2/2m$ and the factor $A(D) = \frac{1}{2^{D-1}\pi^{D/2}\Gamma(D/2)}$.

3 Scalar Curvature

The metric g_{ij} for a thermodynamic system is defined [8] as the second order term after expanding the information distance $I(\rho(\beta^i), \rho(\beta^i + d\beta^i)) = Tr \rho(\ln \rho(\beta^i) - \ln \rho(\beta^i + d\beta^i))$ between two statistical close states $\rho(\beta^i)$ and $\rho(\beta^i + d\beta^i)$ leading to

$$g_{ij} = - \left\langle \frac{\partial^2 \ln \rho}{\partial \beta^i \partial \beta^j} \right\rangle, \quad (7)$$

where in our case the thermodynamic coordinates are $\beta^1 = \beta$ and $\beta^2 = \gamma \equiv -\beta\mu$. For exponential distributions, $\rho = \frac{\exp(-\sum_i \beta^i F^i)}{Z}$, the metric simply reduces to

$$g_{ij} = \frac{\partial^2 \ln Z}{\partial \beta^i \partial \beta^j}. \quad (8)$$

From Eqs. (5) and (8) it is simple to obtain that the metric tensor components are written in terms of the second moments of the energy and particle number as follows

$$\begin{aligned} g_{11} &= \langle H^2 \rangle - \langle H \rangle^2, \\ g_{22} &= \langle N^2 \rangle - \langle N \rangle^2, \\ g_{12} &= \langle NH \rangle - \langle N \rangle \langle H \rangle. \end{aligned} \quad (9)$$

From the metric we obtain the scalar curvature R from the definition

$$R = \frac{2}{\det g} R_{1212}, \quad (10)$$

where $\det g = g_{11}g_{22} - g_{12}g_{12}$ and due to the obvious identities like $\frac{\partial g_{ij}}{\partial \beta^j} \equiv g_{ij,j} = g_{jj,i}$ the non-vanishing part of the curvature tensor R_{ijkl} is given in terms of the Christoffel symbols $\Gamma_{ijk} = \frac{1}{2}g_{ij,k}$ as follows

$$R_{ijkl} = g^{mn} (\Gamma_{mil}\Gamma_{njk} - \Gamma_{mik}\Gamma_{njl}). \quad (11)$$

The calculation of R simply reduces to solve

$$R = \frac{1}{2(\det g)^2} \begin{vmatrix} g_{11} & g_{22} & g_{12} \\ g_{11,1} & g_{22,1} & g_{21,1} \\ g_{11,2} & g_{22,2} & g_{21,2} \end{vmatrix}.$$

where $g_{ij,k} = \partial g_{ij} / \partial \beta^k$. From Eq. (6) we find that the metric components can be written

$$\begin{aligned} g_{11} &= \frac{V}{\beta^2 \lambda^D \Gamma(D/2)} H_{\frac{D}{2}+1} \\ g_{12} &= \frac{V}{\beta \lambda^D \Gamma(D/2)} H_{\frac{D}{2}} \\ g_{22} &= \frac{V}{\lambda^D \Gamma(D/2)} H_{\frac{D}{2}-1}, \end{aligned} \quad (12)$$

where the function H_η is the integral

$$H_\eta = \int_0^\infty dx x^\eta \left(\frac{f'_0}{f_0^2} - \frac{(M+1)f'_M}{f_M^2} \right), \quad (13)$$

where the function $f_L = 1 - z^{L+1} e^{-(L+1)x}$ ($L = 0, 1, \dots, M$) and $f'_L = \partial f_L / \partial \gamma$. The function H_η satisfies the simple relation

$$\frac{\partial H_\eta}{\partial \gamma} = -\eta H_{\eta-1}. \quad (14)$$

We obtain for the scalar curvature R for arbitrary D

$$R = \frac{\lambda^D \Gamma(\frac{D}{2})}{2V} \left[\frac{(1 + \frac{D}{2}) H_{\frac{D}{2}}^2 H_{\frac{D}{2}-1} + (\frac{D}{2} - 1) H_{\frac{D}{2}+1} H_{\frac{D}{2}} H_{\frac{D}{2}-2} - D H_{\frac{D}{2}+1} H_{\frac{D}{2}-1}^2}{(H_{\frac{D}{2}+1} H_{\frac{D}{2}-1} - H_{\frac{D}{2}}^2)^2} \right]. \quad (15)$$

Figures 1 and 2 show the results of a numerical calculation of Equation (15) for the scalar curvature in $D = 3$. Figure 1 shows the dependence of R on the fugacity z for the values $M = 1, 5, 25$. For $M = 5$ the system is bosonic for very low values of z and becomes fermionic for $z > 0.72$. The switch from bosonic to fermionic occurs at higher values of z as M increases. For large values of z and M , these systems are more stable than the fermionic $M = 1$ system. There is a value of z like $z \approx 0.7$ for $M = 5$ and $z \approx 1$ for $M = 25$ where the scalar curvature vanishes and thus M -systems mimic a classical gas behavior. Figure 2 shows the values of the scalar curvature R as a function of $M \geq 1$ for the fugacity values $z = 0.5, 0.9, 1.5, 15$. For $z < 1$, M -systems become bosonic as M increases. For $z > 1$ the behavior is fermionic for all values of M . For any z there is a value of $M > 1$ after which the scalar curvature becomes constant. The instability increases at those values of $z \approx 1$ and larger values of M , as expected. The behavior at $D = 2$, shown in Figures 3 and 4, is quite similar to the one in

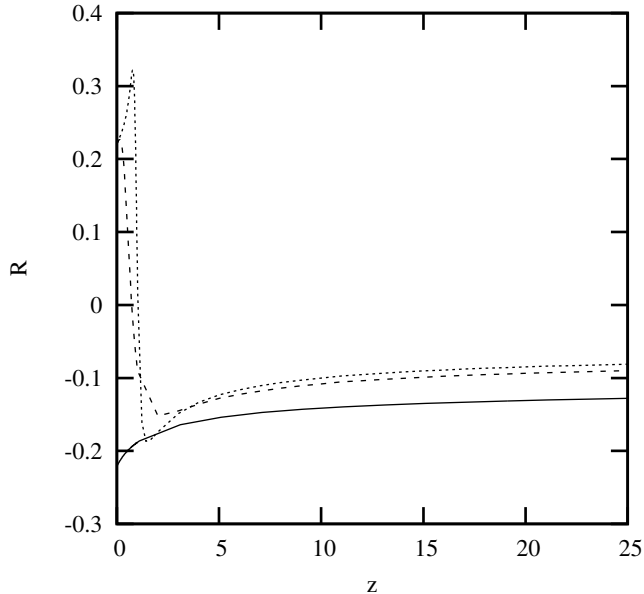


Figure 1: The scalar curvature R , in units of $\lambda^3 V^{-1}$, as a function of the fugacity z at $D = 3$ and constant β for the cases of $M = 1$ (solid line), $M = 5$ (dashed line) and $M = 25$ (dotted line).

three dimensions. Figure 5 displays the scalar curvature for $0 < z < 1$ for the standard Bose-Einstein case $M \rightarrow \infty$, $M = 25$ and $M = 50$. The stability is practically identical up to a value of z where the Bose-Einstein system becomes more unstable approaching the onset of Bose-Einstein condensation and the M -systems are more stable and switching to fermionic ($R < 0$) at higher values of z .

4 Conclusions

In this manuscript we have calculated the scalar curvature for systems following M -statistics in two and three dimensions. Since the scalar curvature R is positive (negative) for bosons (fermions) our results give us a detailed picture about the anyonic behavior than a calculation of the virial coefficients could provide. In addition, the values of the scalar curvature tell us about the correlations and thus the stability of the system. A particular feature of M -systems is that their behavior is very similar in two and three dimensions. For those values $0 < z < 1$ these systems are bosonic and more stable than the standard Bose-Einstein case, and become fermionic at higher values of z . The change from bosonic to fermionic occurs at higher values of z as the maximum occupation number M increases. Therefore, M -statistics provides a different framework, other than quantum group invariant systems [32], to study anyonic behavior if ever observed in either two or three dimensions.

5 Acknowledgment

I thank Y. J. Ng for reading the manuscript and useful comments.

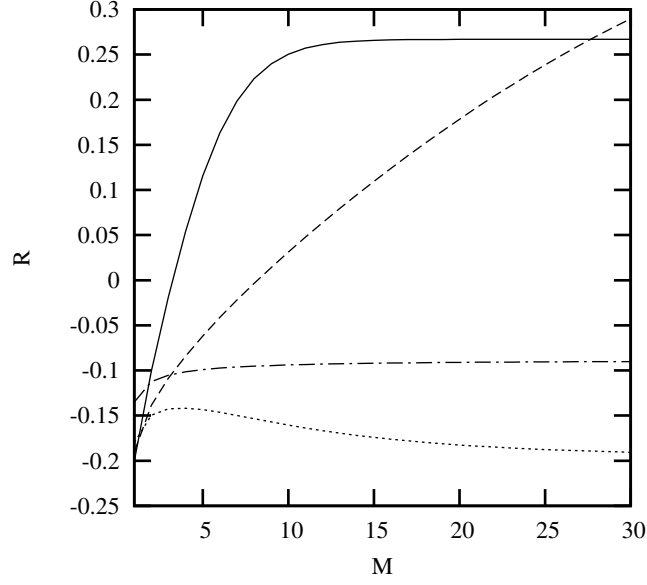


Figure 2: The scalar curvature R at $D = 3$, in units of $\lambda^3 V^{-1}$, as a function of M and values for the fugacity $z = 0.5$ (solid line), $z = 0.9$ (dashed line), $z = 1.5$ (dotted line) and $z = 15$ (dashed-dotted line).

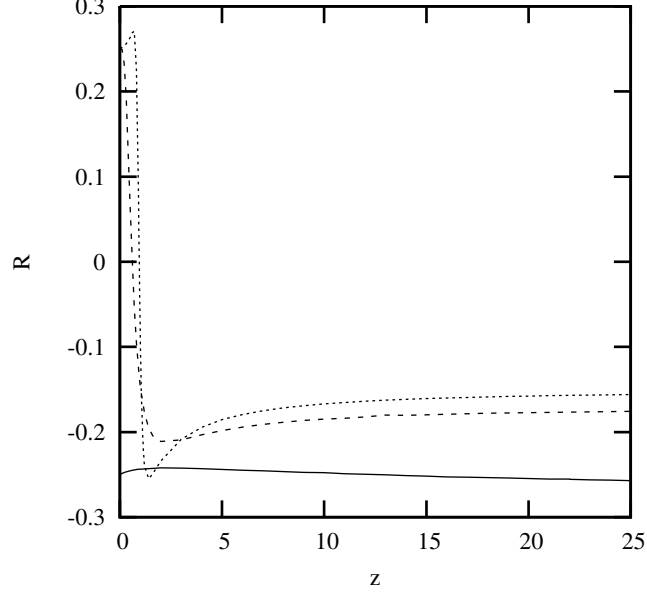


Figure 3: The scalar curvature R , in units of $\lambda^2 A^{-1}$, as a function of the fugacity z at $D = 2$ and constant β for the cases of $M = 1$ (solid line), $M = 5$ (dashed line) and $M = 25$ (dotted line).

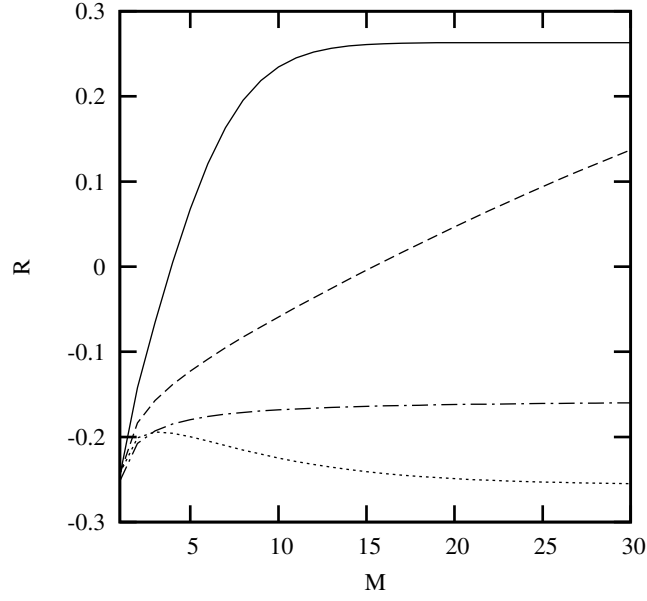


Figure 4: The scalar curvature R at $D = 2$, in units of $\lambda^2 A^{-1}$, as a function of M and values for the fugacity $z = 0.5$ (solid line), $z = 0.9$ (dashed line), $z = 1.5$ (dotted line) and $z = 15$ (dashed-dotted line).

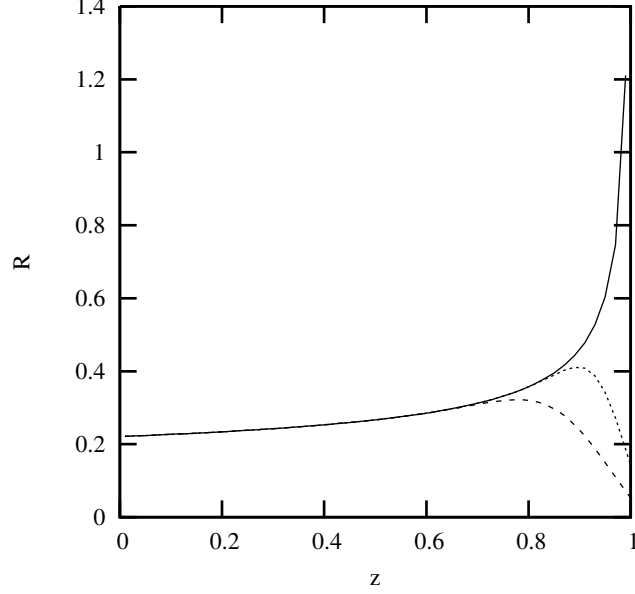


Figure 5: The scalar curvature R , in units of $\lambda^3 V^{-1}$, as a function of the fugacity z , for $0 < z < 1$, at $D = 3$ and constant β for the cases a Bose-Einstein gas ($M \rightarrow \infty$) (solid line), $M = 25$ (dashed line) and $M = 50$ (dotted line).

References

- [1] W. Chen, Y. J. Ng and H. van Dam 1996 Mod. Phys. Lett. A **11** 795 .
- [2] L. Tisza, Generalized Thermodynamics (MIT, Cambridge, 1966).
- [3] R. B. Griffiths and J. C. Wheeler 1970 Phys. Rev. A **2** 1047.
- [4] F. Weinhold 1975 J. Chem. Phys. **63** 2479.
- [5] S.-I. Amari, Differential-Geometrical Methods in Statistics (Springer-Verlag, Berlin, 1985).
- [6] S.-I. Amari and H. Nagaoka, Methods of Information Geometry (AMS, Rhode Island, 2000).
- [7] G. Ruppeiner 1979 Phys. Rev. A **20** 1608.
- [8] R. S. Ingarden, H. Janyszek, A. Kossakowski and T. Kawaguchi 1982 Tensor N.S. **37** 105.
- [9] W. K. Wootters 1981 Phys. Rev. D **23** 357.
- [10] R. Gilmore 1984 Phys. Rev. A **30** 1994.
- [11] G. Ruppeiner 1985 Phys. Rev. A **32** 3141.
- [12] R. Gilmore 1985 Phys. Rev. A **32** 3144.
- [13] J. Nulton and P. Salamon 1985 Phys. Rev. A **31** 2520.
- [14] H. Janyszek 1986 Rep. Math. Phys. **24** 1; Rep. Math. Phys. **24** 11.
- [15] H. Janiszek and R Mrugala 1989 Rep. Math. Phys. **27** 145.
- [16] G. Ruppeiner 1995 Rev. Mod. Phys. **67** 605.
- [17] G. Ruppeiner 2010 Am. J. Phys. **78** 1170 and references therein.
- [18] H. Janiszek and R Mrugala 1990 J. Phys. A: Math. Theor. **23** 467.
- [19] D. Brody and D. Hook 2009 J. Phys. A: Math. Theor. **42** 023001.
- [20] H. Janiszek and R Mrugala 1989 Phys. Rev. A **39** 6515.
- [21] H. Janyszek 1990 J. Phys. A:Math. Theor. **23** 477.
- [22] D. Brody and N. Rivier 1995 Phys. Rev. E **51** 1006.
- [23] W. Janke, D. A. Johnston and R. Kenna 2004 Physica A **336** 181.
- [24] R. Trasarti-Battistoni, cond-mat/0203536.
- [25] M. Portesi, A. Plastino and F. Pennini 2006 Physica A **365** 173.

- [26] A. Ohara 2007 Phys. Lett. A **370** 184.
- [27] B. Mirza and H. Mohammadzadeh 2008 Phys. Rev. E **79** 021127.
- [28] B. Mirza and H. Mohammadzadeh 2009 Phys. Rev. E **80** 011132.
- [29] B. Mirza and H. Mohammadzadeh 2010 Phys. Rev. E **82** 031137.
- [30] B. Mirza and H. Mohammadzadeh 2011 J. Phys. A: Math. Theor. **44** 475003.
- [31] M. R. Ubriaco 2012 Phys. Lett. A **376** 2899.
- [32] M. R. Ubriaco 2012 Phys. Lett. A **376** 3581.